OCR
Oxford Cambridge and RSA

Thursday 20 June 2019 – Morning

A Level Further Mathematics A

Y544/01 Discrete Mathematics

Time allowed: 1 hour 30 minutes

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

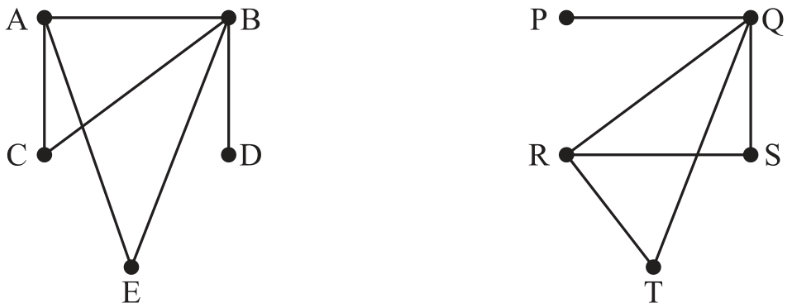
* 7 7 2 3 8 8 0 4 9 *

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 Two graphs are shown below.



Graph G1 Graph G2

(a) **(i)** Prove that the graphs are isomorphic. **[2]**

(ii) Verify that Euler's formula holds for graph G1. **[2]**

(b) Describe how it is possible to add 4 arcs to graph G1 to make a non-planar graph with 5 vertices. **[2]**

(c) Describe how it is possible to add a vertex U and 4 arcs to graph G2 to make a connected non-planar graph with 6 vertices. **[2]**

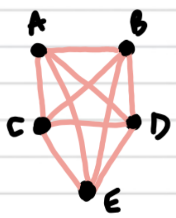
ai) A corresponds with R. (A ⟺ R)
B corresponds with Q. (B ⟺ Q)
C corresponds with S. (C ⟺ S)
D corresponds with P. (D ⟺ P)
E corresponds with T. (E ⟺ T)

(equivalent vertices in 1&2)

aii) $V + R = E + 2$
$5 + R = 6 + 2$
$5 + R = 8$
$R = 3$

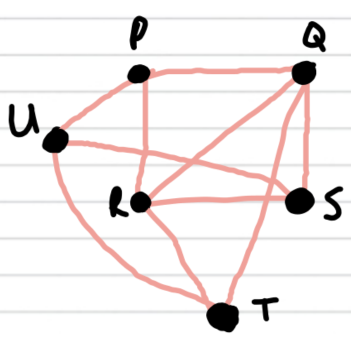
show your working!

Vertices	A, B, C, D, E	V=5
Edges	AB, AC, AE, BC, BD, BE	E=6
Regions	ABC, ABE, ACBE	R=3

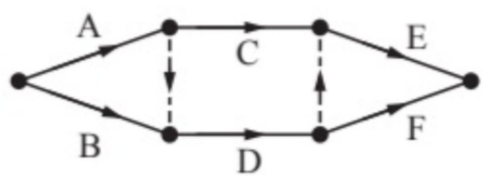


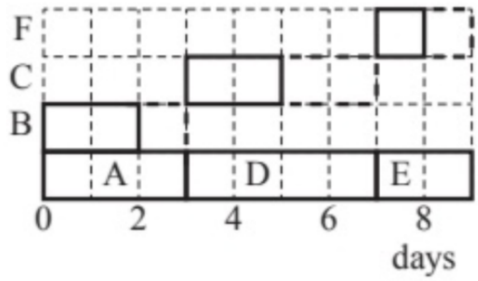
b) Make K_5 by adding arcs AD, CD, CE, DE.

c) Make $K_{3,3}$ by adding vertex U and arcs PR, PU, SU, TU.



2 A project is represented by the activity network and cascade chart below. The table showing the number of workers needed for each activity is incomplete. Each activity needs at least 1 worker.





Activity	Workers
A	2
B	x
C	
D	
E	
F	

(a) Complete the table in the Printed Answer Booklet to show the immediate predecessors for each activity. **[2]**

(b) Calculate the latest start time for each non-critical activity. **[1]**

The minimum number of workers needed is 5.

(c) What type of problem (existence, construction, enumeration or optimisation) is the allocation of a number of workers to the activities? **[1]**

There are 8 workers available who can do activities A and B.

(d) **(i)** Find the number of ways that the workers for activity A can be chosen. **[1]**

(ii) When the workers have been chosen for activity A, find the total number of ways of choosing the workers for activity B for all the different possible values of x, where $x \geqslant 1$. **[2]**

a)

Activity	Immediate predecessors
A	–
B	–
C	A
D	A, B
E	C, D
F	D

b)

Activity	Latest start time (days)
B	1
C	5
F	8

using cascade chart

c) Construction

di) $^8C_2 = 28$ ways to choose 2 workers for A. 8 available, 2 needed $\Rightarrow {}^8C_2$

dii) Number of workers for B can be 1, 2 or 3: min. needed for A & B is 5,

$^6C_1 + {}^6C_2 + {}^6C_3 = 6 + 15 + 20$
$= 41$

5-2=3.

8-2=6 available

if B needs > 3, then minimum workers would be > 5

3 A problem is represented as the initial simplex tableau below.

P	x	y	z	s	t	RHS
1	−2	0	1	0	0	0
0	1	1	1	1	0	60
0	2	3	4	0	1	60

(a) Write the problem as a linear programming formulation in the standard algebraic form with no slack variables. **[3]**

(b) Carry out one iteration of the simplex algorithm. **[3]**

(c) Show algebraically how each row of the tableau found in part **(b)** is calculated. **[3]**

maximise

a) $P_{max} = 2x - z$

subject to:

$x + y + z \leq 60$

$2x + 3y + 4z \leq 60$

$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array}\right\}$ non-negativity

b)

P	x	y	z	s	t	RHS	θ	Row op
1	−2	0	1	0	0	0	$0 \div -2 = 0$	$R1 + 2R3'$
0	1	1	1	1	0	60	$60 \div 1 = 60$	$R2 - R3'$
0	2	3	4	0	1	60	$60 \div 2 = 30$	$R3 \div 2$

new R3

Pivot '2' in column x, row 3.

P	x	y	z	s	t	RHS
1	0	3	5	0	1	60
0	0	−0.5	−1	1	−0.5	30
0	1	1.5	2	0	0.5	30

c) $2x + 3y + 4z + t = 60$

$x = 30 - 1.5y - 2z - 0.5t$

substitute x in $P_{max} - 2x + z = 0$

$P_{max} - (60 - 3y - 4z - t) + z = 0$

$P_{max} + 3y + 5z + t = 60$

$x + y + z + s = 60$

$(30 - 1.5y - 2z - 0.5t) + y + z + s = 60$

$-0.5y - z + s - 0.5t = 30$

4 An algorithm must have an input, an output, be deterministic and finite.

(a) Why is a counter sometimes used in an algorithm? **[1]**

A computer takes 0.2 seconds to sort a list of 500 numbers.

(b) How long would you expect the computer to take to sort a list of 5000 numbers? **[1]**

Simon says that he can sort a list of numbers 'just by looking at them'.

(c) Explain to Simon why sorting algorithms are needed. **[2]**

(d) Demonstrate how quick sort works by using it to sort the following list into **increasing** order. You should indicate the pivots used and which values are already known to be in their correct position.

41 17 8 33 29 **[4]**

For an average case the efficiency of quick sort is $O(n \log n)$, where n is the number of items in the list.

(e) Explain why quick sort is typically quicker than bubble sort and shuttle sort. **[1]**

When the number of comparisons made is used as a measure of the efficiency, the worst case for quick sort is no more efficient than the worst case for bubble sort.

An arrangement of the five numbers from part **(d)** makes up a new list that is to be sorted using the bubble sort or the quick sort.

(f) Without writing out all the passes, determine
- the worst case list
- the total number of comparisons for the worst case list

for each of the algorithms in turn. **[3]**

a) To ensure that the algorithm is finite.

b) $0.2 \times \left(\frac{5000}{500}\right)^2 = 20$ seconds.

c) Practical problems are normally large and cannot be solved efficiently by 'just looking at them' – will take a long time to solve. Algorithms are much more efficient and fast.

d)

$41 \quad 17 \quad 8 \quad 33 \quad 29$

$17 \quad 8 \quad 33 \quad 29 \quad \boxed{41}$

$8 \quad \boxed{17} \quad 33 \quad 29 \quad \boxed{41}$

$\boxed{8} \quad \boxed{17} \quad 29 \quad \boxed{33} \quad \boxed{41}$

$8 \quad 17 \quad 29 \quad 33 \quad 41$

Sort complete.

e) Average case for bubble sort is $O(n\log n) \subset O(n^2)$.

f)

	Worst case	Comparisons
Quick sort	8, 7, 29, 33, 41	10
Bubble sort	41, 33, 29, 17, 8	10

Both use $4+3+2+1 = 10$ comparisons

5 A network is represented by the distance matrix below. For this network a direct connection between two vertices is always shorter than an indirect connection.

	A	B	C	D	E	F	G	H
A	-	130	100	-	-	250	-	-
B	130	-	-	50	-	-	170	100
C	100	-	-	-	80	170	-	90
D	-	50	-	-	-	-	120	-
E	-	-	80	-	-	140	-	120
F	250	-	170	-	140	-	-	-
G	-	170	-	120	-	-	-	90
H	-	100	90	-	120	-	90	-

(a) How does the distance matrix show that the arcs are undirected? **[1]**

The shortest distance from A to E is 180.

(b) Write down the shortest route from A to E. **[1]**

(c) Use Dijkstra's algorithm on the distance matrix to find the length of the shortest route **from G** to each of the other vertices. **[4]**

The arcs represent roads and the weights represent distances in metres. The total length of all the roads is 1610 metres.

Emily and Stephen have set up a company selling ice-creams from a van.

(d) Emily wants to deliver leaflets to the houses along **each side** of each road. Find the length of the shortest continuous route that Emily can use. **[1]**

(e) Stephen wants to drive along each road in the ice-cream van.

(i) Determine the length of the shortest route for Stephen if he starts **at B**. **[5]**

(ii) Stephen wants to use the shortest possible route.

- Find the length of the shortest possible route.
- Write down the start and end vertices of this route. **[2]**

a) Matrix is symmetric about diagonal.

b) A – C – E   from A, can get to C or F. C-E < F-E

c)

Vertex	Temporary Labels	Order of permanent labelling	Permanent label
A	300, 280	7	280
B	170	4	170
C	180	5	180
D	120	3	120
E	210	6	210
F	350	8	350
G	–	1	0
H	90	2	90

d) Length of all roads $= 130 + 100 + 250 + 50 + 80$
(using the table given) $+ 170 + 140 + 100 + 120 + 170$
$+ 90 + 120 + 90$
$= 1610$

$1610 \times 2 = 3220$ m

ei) $AE = 180$ $FG = 350$ $EG = 210$ $EF = 140$
$AF = 250$ 530 460 420
$AG = 280$

Repeat ACHG and EF
$420 + 1610 = 2030$m

eii) Length = 1750 m
Start at A
(or start at G)

6 The pay-off matrix for a game between two players, Sumi and Vlad, is shown below. If Sumi plays A and Vlad plays X then Sumi gets x points and Vlad gets 1 point.

Sumi \ Vlad	X	Y	Z
A	$(x, 1)$	$(4, -2)$	$(2, 0)$
B	$(3, -1)$	$(6, -4)$	$(-1, 3)$

You are given that cell (A, X) is a Nash Equilibrium solution.

(a) Find the range of possible values of x. **[1]**

(b) Explain what the statement 'cell (A, X) is a Nash Equilibrium solution' means for each player. **[2]**

(c) Find a cell where each player gets their maximin pay-off. **[3]**

Suppose, **instead**, that the game can be converted to a zero-sum game.

(d) Determine the optimal strategy for Sumi for the zero-sum game.
- Record the pay-offs for Sumi when the game is converted to a zero-sum game.
- Describe how Sumi should play using this strategy. **[7]**

a) $x \geq 3$

b) Vlad plays X $\Rightarrow$ Sumi's highest score is by playing A.
Sumi plays A $\Rightarrow$ Vlad's highest score is by playing X.

Max $\{x, 3\} = x$
Max $\{1, -2, 0\} = 1$

c)

Sumi \ Vlad	X	Y	Z	min pay-off Sumi
A	$(x, 1)$	$(4, -2)$	$(2, 0)$	2
B	$(3, -1)$	$(6, -4)$	$(-1, 3)$	-1
min pay-off Vlad	-1	-4	0	

play safe for Sumi is A, maximin = 2
" " Vlad is Z, " " = 0

Maximin pay-off for Sumi is 2 and maximin pay-off for Vlad is 0.
Cell (A, Z) has pay-off 2 for Sumi and pay-off 0 for Vlad. $\leftarrow (A, Z) = (2, 0)$

d)

	X	Y	Z	row min.
A	0	3	1	0
B	2	5	-2	-2
column max.	2	5	1	

Game is unstable: $0 \neq 1$

Using the table

Sumi chooses randomly, $P(A) = p$.
- Vlad plays X: $0(p) + 2(1-p) = 2 - 2p$
- Vlad plays Y: $3(p) + 5(1-p) = 5 - 2p$
- Vlad plays Z: $1(p) - 2(1-p) = -2 + 3p$

$2 - 2p = 3p - 2$
$5p = 4 \Rightarrow p = 0.8$
$P(A) = 0.8$ (p)
$P(B) = 0.2$ $(1-p)$

7 Sam is making pies.

There is exactly enough pastry to make 7 large pies or 20 medium pies or 36 small pies, or some mixture of large, medium and small pies.

This is represented as a constraint $180x + 63y + 35z \leqslant 1260$.

(a) Write down what x, y and z represent. **[2]**

There is exactly enough filling to make 5 large pies or 12 medium pies or 18 small pies, or some mixture of large, medium and small pies.

(b) Express this as a constraint of the form $ax + by + cz \leqslant d$, where a, b, c and d are integers. **[2]**

The number of small pies must equal the total number of large and medium pies.

(c) Show that making exactly 9 small pies is inconsistent with the constraints. **[2]**

(d) Determine the maximum number of large pies that can be made.
- Your reasoning should be in the form of words, calculations or algebra.
- You must check that your solution is feasible. **[6]**

a) x = number of large pies made. $1260 \div 7 = 180$, etc.
y = number of medium pies made.
z = number of small pies made.

b) $x : y : z$
$\frac{1}{5} : \frac{1}{12} : \frac{1}{18}$

$36x + 15y + 10z \leq 180$

c) when $z = 9$,
$180x + 63y + 35(9) \leq 1260$
(÷9) $180x + 63y \leq 945$ (÷9)
$\Rightarrow 20x + 7y \leq 105$

$36x + 15y + 10(9) \leq 180$
(÷3) $36x + 15y \leq 90$ (÷3)
$12x + 5y \leq 30$

$x + y = 9$

$5x + 5y = 45$
so
$12x + 5y \leq 30$ cannot be true.

d) Enough filling for 5 large pies:

$x \leq 5$

$180x + 63y + 35z \leq 1260$

$36x + 15y + 10z \leq 180$

$x + y = z$

Eliminate z:

$180x + 63y + 35(x+y) \leq 1260$

$\Rightarrow 215x + 98y \leq 1260$

$36x + 15y + 10(x+y) \leq 180$

$\Rightarrow 46x + 25y \leq 180$

$\frac{180}{46} = 3.913\ldots$

$\therefore \; x \leq 3$

when $x = 3$

$y = 0$ or $y = 1$

$z = 3$ $\quad$ $z = 4$

maximum large pies is 3.

Checking

when $x=3,\ y=0,\ z=3$

$180(3) + 63(0) + 35(3) = 645$

$36(3) + 15(0) + 10(3) = 138$

when $x=3,\ y=1,\ z=4$

$180(3) + 63(1) + 35(4) = 743$

$36(3) + 15(1) + 10(4) = 163$

$645 \leq 1260$

$138 \leq 180$

$743 \leq 1260$

$163 \leq 180$

$\therefore$ Solution satisfies the constraints